# Spatial structure of passive particles with inertia transported by a chaotic flow

Cristóbal López
*Dipartimento di Fisica, Università di Roma "La Sapienza," Piazzale Aldo Moro 2, I-00185, Roma, Italy*
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We study the spatial patterns formed by inertial particles suspended on the surface of a smooth chaotic flow. In addition to the well-known phenomenon of clustering, we show that, in the presence of diffusion and when a steady space-dependent source of particles is considered, the density of particles may show smooth or fractal features in the low density areas. The conditions needed for the appearance of these structures and their characterization with the first order structure function are also calculated.

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In recent years much effort has been given to the study of the spatial patterns that chemically or biologically active particles form when they are advected by chaotic flows [1–6]. Generally, it is supposed that the interplay between the interacting dynamics of the particles and the transport due to the flow gives rise to very complex spatial structures with statistical properties very different from those of passive (in the sense of nonreacting) particles. However, recently have come to light other mechanisms, different from chemical or biological activity, that can also give rise to complex spatial patterns. For example, in [7–11] the inertia of the particles is proposed as a relevant physical property that can produce the aggregation of particles. The inertia induced on the particles by the surrounding fluid perturbs the velocity of the particles with respect to that of the fluid; thus the particles do not strictly follow the fluid [12], and they tend to cluster.

In this work we will study the patterns formed by passive inertial particles transported by a chaotic flow. Apart from the above mentioned phenomenon of clustering, we will show that, when a source of particles is present, fractal or nonfractal structures appear in the spatial areas where the particles are not clustered, which in the following will be called the *background* or *B* set. Moreover, we explicitly write down the conditions for the appearance of these structures and characterize them by calculating the first order structure function.

The main physical assumptions in the rest of this work are as follows. (i) The particles are very small so that the fluid around them is viscous and the Stokes time, which is a measure of the inertia, is small. (ii) The particles form a continuum distribution. (iii) We neglect buoyancy forces. Also, we consider no sedimentation processes and we limit the study to particles driven by a two-dimensional flow. (iv) The flow exhibits *chaotic advection*, that is, there is exponential increase of the distance between two initially close fluid parcels.

Let us consider a small spherical particle of radius $a$ and with density $\rho_p$ suspended in a bidimensional fluid with density $\rho$ and viscosity $\nu$. The velocity of the particle $\mathbf{v}$ is related to the driving fluid velocity $\mathbf{u}$ by [7,12]

$$\mathbf{v}=\mathbf{u}+(\beta-1)\tau\,[\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}]+\Theta(\tau^2), \tag{1}$$

where $\beta=3\rho/(\rho+2\rho_p)$ and $\tau$ is the Stokes time, which is $\tau=a^2/3\beta\nu$. Note that for particles heavier than the fluid $\tau_p\equiv(\beta-1)\tau$ is negative, and when they are lighter $\tau_p$ is positive. The sign of $\tau_p$ is not relevant in this work as we are considering that the particles are moving on the surface of the fluid.

The important fact is that, although we assume that the fluid flow is incompressible $\boldsymbol{\nabla}\cdot\mathbf{u}=0$, the particle velocity, however, is compressible, $\boldsymbol{\nabla}\cdot\mathbf{v}=\tau_p\boldsymbol{\nabla}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\neq 0$.

The evolution of the number density or concentration of particles $n(\mathbf{x},t)$ is given by

$$\frac{\partial n(\mathbf{x},t)}{\partial t}+\boldsymbol{\nabla}\cdot[n(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)]=S(\mathbf{x})+D\nabla^2 n(\mathbf{x},t), \tag{2}$$

where $D$ is the diffusivity of the particles, coming from their Brownian motion at very small length scales, and $S(\mathbf{x})$ is a stationary source of particles with zero spatial mean. The field $n$ has been assumed to have no back influence on the velocity $\mathbf{v}$. Thus, after reading Eq. (2) it is worth mentioning that this work is concerned with the study of the spatial patterns of a forced passive scalar advected by a compressible two-dimensional (2D) chaotic flow.

Equation (2) is rewritten as

$$\frac{\partial n(\mathbf{x},t)}{\partial t}+(\mathbf{v}\cdot\nabla)n(\mathbf{x},t)=S(\mathbf{x})+D\nabla^2 n-n\boldsymbol{\nabla}\cdot\mathbf{v}, \tag{3}$$

which can be solved using the so-called Feynmann-Kac or stochastic Lagrangian representation (assuming a zero initial condition) [13]:

$$n(\mathbf{x},t)=\left\langle \int_0^t ds S[\mathbf{r}(s)]e^{-\int_s^t\boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{r}(s'))ds'}\right\rangle_{\boldsymbol{\eta}}, \tag{4}$$

where $\mathbf{r}(s)$ is the solution of

$$\frac{d\mathbf{r}}{ds}=\mathbf{v}(\mathbf{r}(s),s)+\sqrt{2D}\,\boldsymbol{\eta} \tag{5}$$

that satisfies the final condition $\mathbf{r}(t)=\mathbf{x}$. $\boldsymbol{\eta}(s)$ is a normalized vector-valued white noise term with zero mean, i.e., $\langle\boldsymbol{\eta}(s)\rangle_{\boldsymbol{\eta}}=0$ and $\langle\boldsymbol{\eta}(s)\boldsymbol{\eta}(s')\rangle_{\boldsymbol{\eta}}=\mathbf{I}\delta(s-s')$, with $\mathbf{I}$ the identity matrix. The average $\langle\cdot\rangle_{\boldsymbol{\eta}}$ is taken over the different stochastic trajectories $\mathbf{r}(s)$ ending at the stated final point $\mathbf{x}$.

Equation (4) relates a Eulerian quantity, the $n(\mathbf{x},t)$ field, with a Lagrangian one, the trajectory of any particle $\mathbf{r}(s)$

 

ending at $\mathbf{x}$ at time $t$. Because of the zero spatial mean of the source, indicating that there are sources and sinks of particles, the total concentration of particles is constant. Moreover, in the long time limit the concentration field approaches a state with the same time dependence as the flow of particles [7], which is the same as that of the fluid flow. Thus, if the given velocity field is periodic in time, the concentration field approaches a periodic steady field. Nonetheless, the spatial statistical properties of $n(\mathbf{x},t)$ remain constant for large times.

In the absence of the source term the physical mechanisms of clustering are well understood [7,8]. The particles aggregate on those spatial points where $b(\mathbf{x},t,t_0)\equiv[1/(t-t_0)]\int_{t_0}^{t}\nabla\cdot\mathbf{v}(\mathbf{r}(s),s)ds<0$ with $t_0=0$. These constitute a fractal set which in the following will be called the *clustering* or $C$ attractor. Diffusion acts to stop the exponential growth of concentration on this fractal set.

With the source term the above mentioned scenario is maintained, but now the injection and subtraction of particles may or may not form fractal patterns in the background set. To see this we consider the behavior of the spatial gradients of the field $n(\mathbf{x},t)$, and we keep in mind that $n$ is fractal where it is a singular function of $\mathbf{x}$. For simplicity, we first take the large Schmidt number limit $\nu/D$, and neglect diffusion, i.e., we neglect the average in Eq. (4). We calculate (formally) the difference of $n$ for two infinitesimally close particle points, $\delta n=n[\mathbf{r}(s)+\delta\mathbf{r}(s),s]-n[\mathbf{r}(s),s]$, by integrating backward in time the Lagrangian trajectory of two particles finishing at time $t$ in the positions $\mathbf{x}+\delta\mathbf{x}$ and $\mathbf{x}$, respectively. Then we come back to the properties of the fluid and we note that, in particular, the trajectories of the fluid parcels separate exponentially with time. The nonlinear relation between fluid velocity and particle velocity indicates that the trajectories of the particles also diverge exponentially for long times, that is, $|\delta\mathbf{r}(s)|\sim|\delta\mathbf{r}(0)|\exp(\lambda s)$, for nearly all the initial orientations of the initial particle separation $\delta\mathbf{r}(0)$. The positive number $\lambda$ is the maximum Lyapunov exponent associated with the chaotic trajectories of the particles as driven by the field $\mathbf{v}$ [14]. In principle, $\lambda$ depends both on the initial condition and on the duration of the trajectory. However, in the long-time limit $\lambda$ is the same for almost all trajectories, but deviations can persist on fractal sets of measure zero [14]. These can give rise to multifractal corrections [4] that will not be considered in the present work. Therefore, in the following we assume that $\lambda$ takes its most probable value and we neglect its spatial or time dependence.

Finally, the gradient of the field at a given spatiotemporal point $(\mathbf{x},t)$ is

$$\begin{aligned}\mathbf{m}\cdot\nabla n(\mathbf{x},t)\approx\mathbf{m}\cdot&\left(\int_0^t ds\,\nabla S[\mathbf{r}(s)]e^{[\lambda-b(\mathbf{x},t,s)](t-s)}\right)\\&-\mathbf{m}\cdot\left(\int_0^t ds\,S[\mathbf{r}(s)]e^{[\lambda-b(\mathbf{x},t,s)](t-s)}\right.\\&\left.\times\int_s^t dw\,\nabla(\nabla\cdot\mathbf{v})e^{\lambda(s-w)}\right).\end{aligned}\tag{6}$$

Here $\mathbf{m}$ is a unit vector in the direction of the final $\delta\mathbf{r}(t)$.

In the large time limit, $t\to\infty$, we can approximate $b(\mathbf{x},t\to\infty,s)\approx b(\mathbf{x},\infty,0)\equiv b(\mathbf{x})$ because it dominates in the time integral. Thus it is clear from Eq. (6) that, for long times, the gradient of the field $n(\mathbf{x},t\to\infty)$ depends critically on the sign of the exponential $\lambda-b(\mathbf{x})$, in such a way that when in a spatial area of the background $\lambda-b(\mathbf{x})>0$ the gradients are not bounded, i.e., $n$ is a nonhomogenous rough field in that spatial zone. In contrast, in those areas where $\lambda-b(\mathbf{x})<0$ the density field is smooth. Concerning the clustering areas, as $b(\mathbf{x})$ is always negative, the sign of the exponential is positive, and they are always rough.

It is important to note that the dynamics of a passive scalar advected by a compressible flow, as given by Eq. (3), is rather analogous to that of an advected passive scalar with finite lifetime [4,6,15]. In fact, the above calculations closely follow those shown in [4]. The analogy is that the last term in Eq. (3) acts as a decaying term at all spatial points but in the clustering attractor. The main difference comes from the negative values of $b(\mathbf{x})$ in this attractor.

Following this analogy we can characterize the fractal patterns of the $n$ field. We first neglect the spatial dependence of $b(\mathbf{x})$ by performing spatial averages [6] in every set where it takes a constant sign, that is, in the $B$ and $C$ sets. Thus, we denote as $b^C=\langle b(\mathbf{x})\rangle_C$, and $b^B=\langle b(\mathbf{x})\rangle_B$ the spatial averages of $b(\mathbf{x})$ over the $C$ set and over the $B$ set, respectively. It is clear that $b^C<0$ and $b^B>0$. In the background the correspondence with [4,6] is straightforward and we have that $\delta n^B\sim\delta r^{\alpha^B}$, with $\delta n^B$ denoting $\delta n$ in the $B$ set, and $\delta r\equiv|\delta\mathbf{r}(t)|$. From [4] $\alpha^B=\min(1,b^B/\lambda)$. In the $C$ set this correspondence is not so obvious; however, from Eq. (6) $\delta n^C\sim\delta r e^{(\lambda-b^C)}$ and the fact that in the backward-in-time dynamics $\delta r\sim|\delta\mathbf{r}(0)|e^{-\lambda t}$, we obtain that $\delta n^C\sim\delta r^{\alpha^C}$ with $\alpha^C=b^C/\lambda$. This expression is valid for small scales larger than the typical diffusion length scale, $L_d\sim\sqrt{2D/\lambda}$, and shows that in the clustering set $\delta n$ grows when the length scale decreases.

A proper characterization of the spatial structure, accessible to experiments, is the first order structure function, i.e., the spatial average of $\delta n$ along a one-dimensional spatial cut (taken of unit length for simplicity), $\langle\delta n\rangle$. Obviously $\langle\delta n\rangle=\langle\delta n^C\rangle_C+\langle\delta n^B\rangle_B$. Along the transect, the number of segments of length $\delta r$ is $1/\delta r$, and the number of segments containing parts of the clustering set is $\delta r^{-D_0+1}$, $D_0$ being the fractal dimension of this attractor. We obtain

$$\langle\delta n\rangle=\delta r(\delta r)^{-D_0+1}\delta r^{\alpha^C}+\delta r[(\delta r)^{-1}-(\delta r)^{-D_0+1}]\delta r^{\alpha^B}.\tag{7}$$

Therefore, $\langle\delta n\rangle$ scales as $\delta r^{\alpha}$ for small $\delta r$ (but larger than $L_d$) with $\alpha=\min(2-D_0+b^C/\lambda,\alpha^B)=\min(2-D_0+b^C/\lambda,\min(1,b^B/\lambda))$. More specifically, we obtain the result that when the background is smooth then $\alpha=1$ if $D_0<1+b^C/\lambda$ (that is, if the clustering attractor is thin enough), and $\alpha=2-D_0+b^C/\lambda$ if $D_0>1+b^C/\lambda$. On the contrary, when the background is rough $\alpha=b^B/\lambda$ when $D_0<1+b^C/\lambda-b^B/\lambda$, and $\alpha=2-D_0+b^C/\lambda$ in the opposite case. In conclusion, when $D_0$ is small enough $\delta n$ scales as in the $B$ set;

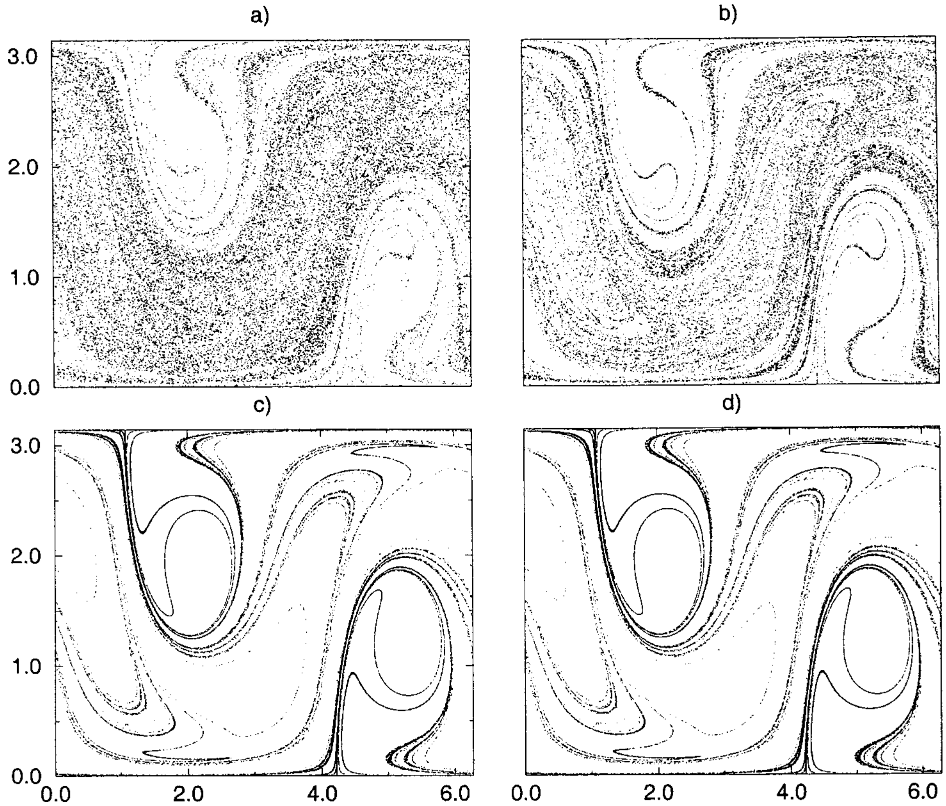


FIG. 1. Clustering attractor when the final time is 100 units. (a) $\tau_p = -0.005$, (b) $\tau_p = -0.01$, and (c) $\tau_p = -0.03$. (d) shows the spatial points where $b(100,0) < 0$ ($\tau_p = -0.03$).

however, when it is not too small the existence of the clustering set strongly changes the scaling of $\delta n$.

To check our analytical predictions we performed numerical simulations with the simple 2D model flow defined by the stream function

$$\psi(x,y,t) = A \sin(x + B \cos wt) \sin y, \tag{8}$$

which, due to the time dependence, can show chaotic advection. $A, B$, and $w$ are real positive parameters, and periodic boundary conditions are used. The equations of motion for an element of fluid are given by $\dot{x} = \partial_y \psi$ and $\dot{y} = -\partial_x \psi$. In the following we always use $A = 10$, $B = 10$, and $w = 1$. We also take negative values of $\tau_p$ in our simulations, but similar results are obtained when it is positive. For the source term we use $S(x,y) = \sin(x)\cos(y)$.

First, in Figs. 1(a), 1(b), and 1(c), we show the clustering attractor (when the final time is 100 units) for $\tau_p = -0.005$, $-0.01$, and $-0.03$. It is calculated by evolving many particles, which initially are uniformly distributed in the interval $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant \pi$, with the dynamical system $d\mathbf{r}/dt = \mathbf{v}$. Also in Fig. 1(d) we plot the spatial points where the $b(t = 100,0)$ is negative for $\tau_p = -0.03$, and we can see that, as expected, the result is the same as the clustering attractor in Fig. 1(c). The numerical values of the fractal dimensions of the $C$ set and of the Lyapunov exponents for the flow of particles are shown in Table I for these values of the $\tau_p$ parameter. From these numbers we can observe that as the absolute value of $\tau_p$ becomes larger, the smaller is the fractal dimension of the $C$ set, i.e., it is *more fractal*. For very small values of $\tau_p$ the particles are almost distributed through the whole system.

In order to calculate transects of the $n$ field we integrate Eq. (5) backward in time with initial conditions on a one-dimensional cut with $y = 1$ and $0 \leqslant x \leqslant 2\pi$. Many different trajectories are obtained for different realizations of the noise. Then we calculate the scalar field for any of these trajectories following Eq. (4) and, finally, we perform the averaging. In this way the values of $b^C$ and $b^B$ in the transect are calculated and displayed in Table I. In Fig. 2 these transects are shown for a value of $D = 2 \times 10^{-4}$ and $\tau_p = -0.005, -0.01, -0.03$. Figure 2(a) is for a very small $\tau_p = -0.005$, so the effect of inertia is almost negligible and particles are distributed through almost the whole phase space, with no significant peaks in the concentration. Moreover, the spatial structure is always fractal as this case should correspond with the passive scalar. For $\tau_p = -0.01$, Fig. 2(b), there appear peaks in the concentration which correspond to the clustering attractor. Between peaks one can also observe the rough structure of the background set. This is because $\lambda \sim 0.60 > b^B \sim 0.34$. In contrast, in Fig. 2(c), obtained for $\tau_p = -0.03$, the background areas are always smooth because $\lambda \sim 0.92 < b^B \sim 2.07$.

TABLE I. $b^B$ and $b^C$ fractal dimensions and Lyapunov exponents for different values of $\tau_p$.

| $\tau_p$ | $b^B$ | $b^C$ | $D_0$ | $\lambda$ |
|---|---|---|---|---|
| $-0.005$ | 0.137(9) | $-0.015(2)$ | 1.95(5) | 0.58(8) |
| $-0.01$ | 0.347(7) | $-0.018(7)$ | 1.91(9) | 0.60(2) |
| $-0.03$ | 2.073(4) | $-0.177(9)$ | 1.51(2) | 0.91(9) |

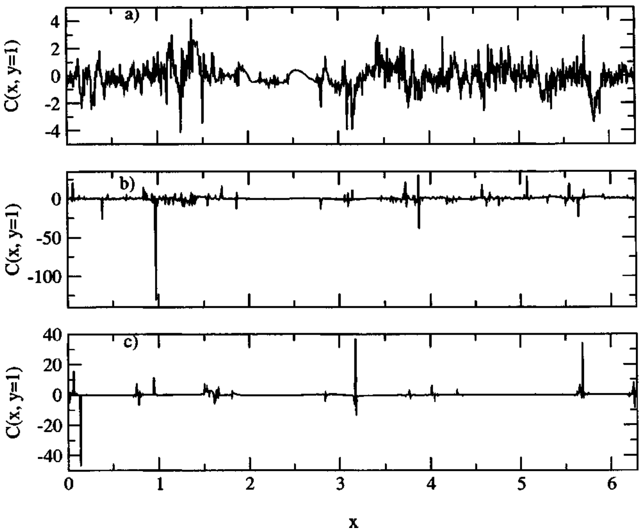


FIG. 2. Transects of the density field along the line $y=1$. (a) $\tau_p=-0.005$, (b) $\tau_p=-0.01$, and (c) $\tau_p=-0.03$. Other parameters used are $A=10$, $B=10$, $w=1$, and $D=2\times10^{-4}$.

Finally, we calculate the first order structure function $\langle \delta n \rangle$ for the transects of Fig. 2. This is shown in Fig. 3, where we also plot a straight line with the slope $\alpha$ given in the discussion below Eq. (7). The agreement is rather accurate for scales larger than $L_d \sim 0.025$.

Summing up, we have shown that a very complex spatial distribution may arise when particles with inertia are transported by a chaotic flow. The inertia induces the aggregation of particles in the so-called clustering set, diffusion stops the growing without bound of the density of particles in this set, and then the presence of the source term gives rise to the formation of inhomogeneous structures in the background set. We have shown that these structures may show smooth or fractal features depending on the relation between the Lyapunov exponents of the chaotic flow of the particles and the values of the field $b(\mathbf{x},t,0)=(1/t)\int_0^t \boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{r}(s),s)ds$, in the long-time limit. Also, the characterization of the spatial structures of the particles has been performed with the first order spatial structure function, and we have found that it scales with an exponent that depends on the fractal dimension of the $C$ set, the maximum Lyapunov exponent of the particles flow, and the averages values of $b(\mathbf{x})$ over the $C$ and $B$ sets.

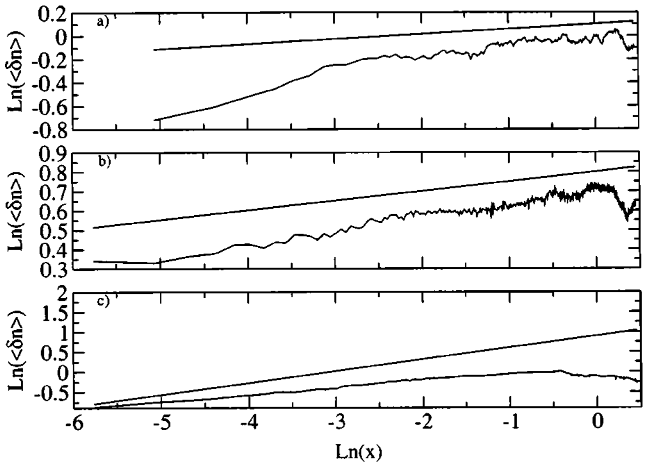


FIG. 3. Structure functions for the corresponding transects in Fig. 2. The solid line has the slope given by the theoretical prediction $\alpha=\min(2-D_0+b^C/\lambda,\min(1,b^B/\lambda))$.

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